

8.333 Fall 2025 Recitations 6b: Large- N tricks

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October 29, 2025

These notes are largely a conglomeration of the previous years' recitation notes by Julien Tailleur, Amer Al-Hiyasat, and Sara Dal Cengio.

See also Sec. 2.6-2.7 of Mehran Kardar's *Statistical Physics of Particles*.

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1 Saddle point integral

Suppose there is a function $f : \mathbb{R} \rightarrow \mathbb{R}$ with a global minimum at x_0 . Suppose also that N is a very large positive number, and $g : \mathbb{R} \rightarrow \mathbb{R}$ is another arbitrary function. Then, we can approximate the integral

$$\int_{-\infty}^{\infty} dx e^{-Nf(x)} g(x) \approx \int_{-\infty}^{\infty} dx e^{-Nf(x_0) - N\frac{1}{2}(x-x_0)^2 f''(x_0)} \left[g(x_0) + \frac{1}{2}(x-x_0)^2 g''(x_0) + \dots \right] \quad (1)$$

$$= e^{-Nf(x_0)} \sqrt{\frac{2\pi}{Nf''(x_0)}} \left[g(x_0) + \mathcal{O}(1/N) \right]. \quad (2)$$

Thus, we have the following theorem:

Theorem 1.1: Saddle point approximation

Consider $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that f has a global maximum at $x = x_0$. Suppose N is a positive number. Then, the following holds in the large- N limit:

$$\int_{-\infty}^{\infty} dx e^{-Nf(x)} g(x) \xrightarrow{N \rightarrow \infty} e^{-Nf(x_0)} \sqrt{\frac{2\pi}{Nf''(x_0)}} \left[g(x_0) + \mathcal{O}(1/N) \right]. \quad (3)$$

Example 1.1: A δ -function is the limit of narrowing Gaussians

We will use the saddle point approximation to justify the statement that “the limit of narrowing Gaussian functions is a δ -function”. Recall that the δ function is *defined by* the property that, for any function $g : \mathbb{R} \rightarrow \mathbb{R}$,

$$\int_{-\infty}^{\infty} \delta(x - x_0)g(x)dx = g(x_0) . \quad (4)$$

Now, consider the Gaussian distribution

$$\rho_\varepsilon(x - x_0) = \frac{e^{-(x-x_0)/2\varepsilon}}{\sqrt{2\pi\varepsilon}} . \quad (5)$$

As $\varepsilon \rightarrow 0$, $\rho_\varepsilon(x - x_0)$ approaches a narrow function peaked at x_0 but with an integral of 1. For a function g , let's calculate the integral

$$\int_{-\infty}^{\infty} g(x)\rho_\varepsilon(x - x_0)dx = \int_{-\infty}^{\infty} \frac{e^{-(x-x_0)/2\varepsilon}}{\sqrt{2\pi\varepsilon}}g(x) . \quad (6)$$

Now we will use the saddle-point approximation (3) in the $\varepsilon \rightarrow 0$ limit. We find

$$\int_{-\infty}^{\infty} g(x)\rho_\varepsilon(x - x_0)dx \xrightarrow{\varepsilon \rightarrow 0} g(x_0) + \mathcal{O}(\varepsilon) . \quad (7)$$

Thus, to leading order in ε , $\rho_\varepsilon(x - x_0)$ approximates $\delta(x - x_0)$.

2 Stirling's approximation

In this section, we will prove Stirling's approximation (given without proof in recitation 3) using the saddle point approximation.

For a very large N , we can approximate its factorial using the following:

$$N! = \int_0^\infty x^N e^{-x} dx = \int_0^\infty e^{-x+N \ln x} dx = e^{N \ln N} \int_0^\infty e^{-N(y - \ln y)} . \quad (8)$$

The first equality can be seen by integrating by parts N times. In the final equality, we have changed variables from x to $y = Nx$. Note, then, that the function

$$f(y) \equiv y - \ln y \quad (9)$$

has a global minimum at $y = 1$ of $f(1) = 1$, with $f''(1) = 1$. Thus, we can use the saddle point approximation (3) to write

$$N! \approx e^{N \ln N - N} \sqrt{\frac{2\pi}{N}} . \quad (10)$$

Thus, we have Stirling's approximation:

Theorem 2.1: Stirling's approximation

Suppose N is a large integer. Its factorial can be approximated by

$$\ln N! = N \ln N - N + \mathcal{O}(\ln N) . \quad (11)$$

One useful, common application of this is for calculating the log of the binomial coefficient:

Example 2.1: Stirling's approximation for the binomial coefficient

Suppose N is a very large integer, and $n \leq N$ is also large. Define $x \equiv n/N$. Then, we can approximate the binomial coefficient by

$$\ln \binom{N}{n} = \ln N! - \ln(N-n)! - \ln n! \quad (12)$$

$$\approx N \ln N - (N-n) \ln(N-n) - n \ln n \quad (13)$$

$$= N \left[\ln N - (1-x) (\ln(1-x) + \ln N) - x (\ln x + \ln N) \right] \quad (14)$$

$$= -N [x \ln x + (1-x) \ln(1-x)] . \quad (15)$$

3 Central limit theorem

(For more context, see recitation 2.) Consider random variables x_i which are independent, i.e. $\rho(\vec{x}) = \prod_{i=1}^N \rho_i(x_i)$, with characteristic function $\tilde{\rho}(\vec{k}) = \prod_{i=1}^N \tilde{\rho}_i(k_i)$. Then, cumulants linking different variables are zero, and the cumulants of their sum $X = \sum_{i=1}^N x_i$ reduce to

$$x_i \text{ independent} \implies \langle X^n \rangle_c = \sum_{i=1}^N \langle x_i^n \rangle_c . \quad (16)$$

Suppose the variables x_i are, moreover, identically distributed. So, we say they are **independent and identically distributed (iid)**. Then, we have

$$x_i \text{ iid} \implies \langle X^n \rangle_c = N \langle x_i^n \rangle_c . \quad (17)$$

This generalizes the result we found for the binomial distribution: The mean and variance are both proportional to N . Consider a new random variable, Y :

$$Y \equiv \frac{X - \langle X \rangle_c}{\sqrt{N}} . \quad (18)$$

All cumulants of Y scale like N , and therefore the n th cumulant of Y scales like $N/(\sqrt{N})^n = N^{1-n/2}$. In the limit as $N \rightarrow \infty$, only the variance remains; all other cumulants approach zero. But the only probability distribution with zero cumulants above 2 is the normal distribution! We have just proven a weaker version of the **central limit theorem**:

Theorem 3.1: Central limit theorem

Consider N identically distributed random variables $\{x_i\}$. Suppose their correlations are weak enough so that

$$\sum_{i_1=1}^N \dots \sum_{i_m=1}^N \langle x_{i_1} * \dots * x_{i_m} \rangle_c \ll \mathcal{O}(N^{m/2}) . \quad (19)$$

Then, the probability density of their sum approaches a Gaussian distribution:

$$\lim_{N \rightarrow \infty} \rho \left(y \equiv \frac{\sum_{i=1}^N x_i - N \langle x \rangle_c}{\sqrt{N}} \right) = \frac{1}{\sqrt{2\pi \langle x^2 \rangle_c}} \exp \left[-\frac{y^2}{2 \langle x^2 \rangle_c} \right] . \quad (20)$$

Here's a toy example. We will show the convergence to a Gaussian both with and without use of the CLT.

Example 3.1: Biased coin flips

Let i range from 1 to N . Let x_i be a random variable that takes on values $\{0, 1\}$. Suppose $P(0) = 1 - q$ and $P(1) = q$, and consider the random variables

$$X \equiv \sum_{i=1}^N x_i, \quad Y \equiv \frac{X - \langle X \rangle_c}{\sqrt{N}}. \quad (21)$$

- **Without using the central limit theorem**, we can explicitly calculate the probability measure P_X of X . This is simply

$$P_X(n) = \binom{N}{n} (1-q)^{N-n} q^n = \frac{N!}{n!(N-n)!} (1-q)^{N-n} q^n \quad (22)$$

for integers $n \in \{0, 1, \dots, N\}$. Now suppose N is very large. Define $x \equiv n/N$. For n which are $\mathcal{O}(N)$, we can use Example 2.1 to approximate this as

$$P_X(xN) \approx \exp \left[-N \left(x \ln \left(\frac{x}{q} \right) + (1-x) \ln \left(\frac{1-x}{1-q} \right) \right) \right] \equiv \exp [-N f_q(x)]. \quad (23)$$

The function $f_q(x)$ has a minimum at $x_0 = q$, where $f(x_0) = 0$ and $f''(x_0) = \frac{1}{q} + \frac{1}{1-q}$. Thus, using the saddle point approximation (which can be straightforwardly extended to the complex numbers), we may calculate the moment generating function:

$$\langle e^{-ikx} \rangle = \int_0^1 dx e^{-N f_q(x) - ikx} \approx \int_{-\infty}^{\infty} dx e^{-\frac{N}{2} (x-q)^2 - ikx} = \sqrt{\frac{2\pi}{N f_q''(q)}} \exp \left[-\frac{k^2}{2N f_q''(q)} - ikq \right] \quad (24)$$

$$= \sqrt{\frac{2\pi q(1-q)}{N}} \exp \left[-\frac{k^2}{2} \frac{q(1-q)}{N} - ikq \right]. \quad (25)$$

Thus, the cumulant generating function is

$$\ln \langle e^{-ikx} \rangle = -\frac{k^2}{2} \frac{q(1-q)}{N} - ikq - \frac{1}{2} \ln \left(\frac{2\pi q(1-q)}{N} \right), \quad (26)$$

which is a 2nd-degree polynomial in k . The only such distribution is the Gaussian distribution with mean $\mu = q$ and variance $\sigma^2 = q(1-q)/N$. Thus, $X = Nx$ is a Gaussian with mean Nq and variance $Nq(1-q)$.

- **Using the central limit theorem**, we can simply use the fact that X is a sum of N independent random variables, each with mean q and variance $q(1-q)$, to find

$$\rho_Y \left(Y = \frac{X - \langle X \rangle_c}{\sqrt{N}} \right) = \frac{1}{\sqrt{2\pi q(1-q)}} \exp \left(-\frac{Y^2}{2q(1-q)} \right) \quad (27)$$

$$\implies \rho_X(X) = \frac{1}{\sqrt{N}} \rho_Y(Y(X)) = \frac{1}{\sqrt{2\pi Nq(1-q)}} \exp \left(-\frac{(X - Nq)^2}{2Nq(1-q)} \right). \quad (28)$$

This is the same as the result we found without using the CLT.